

THE SNAKE LEMMA

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The snake lemma in abelian categories is a simple and very useful result; in the following, we will present a version of the snake lemma that contains the usual formulation as a special case.

Theorem 1 (Snake Lemma). *Assume that*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

is a commutative diagram of abelian groups with exact rows. Then there exists a homomorphism $\delta : \ker \gamma \cap \operatorname{im} g \rightarrow A' / (\operatorname{im} \alpha + \ker f')$ such that the following sequence is exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker f & \longrightarrow & \ker f' \circ \alpha & \longrightarrow & \ker \beta \longrightarrow \ker \gamma \cap \operatorname{im} g \\ & & & & & & \delta \downarrow \\ 0 & \longleftarrow & \operatorname{coker} g' & \longleftarrow & \operatorname{coker} \gamma \circ g & \longleftarrow & \operatorname{coker} \beta \longleftarrow A' / (\operatorname{im} \alpha + \ker f') \end{array}$$

If f' is injective, then $\ker f' \circ \alpha = \ker \alpha$ and $A' / (\operatorname{im} \alpha + \ker f') = \operatorname{coker} \alpha$; if g is surjective, then $\operatorname{coker} \gamma \circ g = \operatorname{coker} \gamma$ and $\ker \gamma \cap \operatorname{im} g = \ker \gamma$. Thus if f' is injective and g is surjective, then we get the following exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker f & \longrightarrow & \ker \alpha & \longrightarrow & \ker \beta \longrightarrow \ker \gamma \\ & & & & & & \delta \downarrow \\ 0 & \longleftarrow & \operatorname{coker} g' & \longleftarrow & \operatorname{coker} \gamma & \longleftarrow & \operatorname{coker} \beta \longleftarrow \operatorname{coker} \alpha \end{array}$$

The proof of the standard version of the snake lemma goes through.

Corollary 2 (Ring Lemma). *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ be homomorphisms; then there is an exact sequence*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker \alpha & \longrightarrow & \ker(\beta \circ \alpha) & \longrightarrow & \ker \beta \\ & & & & & & \downarrow \\ 0 & \longleftarrow & \operatorname{coker} \beta & \longleftarrow & \operatorname{coker}(\beta \circ \alpha) & \longleftarrow & \operatorname{coker} \alpha \end{array}$$

Proof. Apply the snake lemma to the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \longrightarrow & \operatorname{coker} \alpha & \longrightarrow & 0 \\ \downarrow \beta \circ \alpha & & \downarrow \beta & & \downarrow & & \\ 0 & \longrightarrow & C & \xrightarrow{\operatorname{id}} & C & \longrightarrow & 0 \end{array}$$

□

Bass has observed that the 6-term exact sequence fits into the following exact and commutative diagram (the exact ring):

$$\begin{array}{ccccccc}
 & & \ker \beta & \longrightarrow & \operatorname{coker} \alpha & & \\
 & \nearrow & \searrow & & \nearrow & \searrow & \\
 & & B & & & & \\
 & \nearrow & \searrow & & \nearrow & \searrow & \\
 \ker \beta \circ \alpha & \xrightarrow{\quad} & A & \longrightarrow & C & \longrightarrow & \operatorname{coker} \beta \circ \alpha \\
 & \nwarrow & \nearrow & & \nwarrow & \nearrow & \\
 & & \ker \alpha & \longleftarrow & 0 & \longleftarrow & \operatorname{coker} \beta
 \end{array}$$

Corollary 3 (4-Lemma). *Assume that the diagram*

$$\begin{array}{ccccccc}
 A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\
 \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow \\
 A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D'
 \end{array}$$

is commutative with exact rows. If α is surjective and if δ is injective, then we have the following exact sequences

$$0 \longrightarrow \ker \beta \cap \ker g \longrightarrow \ker \beta \xrightarrow{g^*} \ker \gamma \longrightarrow 0, \quad (1)$$

$$0 \longrightarrow \operatorname{coker} \beta \xrightarrow{g'_*} \operatorname{coker} \gamma \longrightarrow C' / (\operatorname{im} \gamma + \operatorname{im} g') \longrightarrow 0. \quad (2)$$

In particular, we have $\ker \gamma = g(\ker \beta)$ and $\operatorname{im} \beta = g'^{-1}(\operatorname{im} \gamma)$.

Proof. Apply the snake lemma to the diagram consisting of the second and the third square; observing that $\ker \delta = 0$ provides us with the exact sequence

$$0 \longrightarrow \ker g \xrightarrow{\iota} \ker g' \circ \beta \xrightarrow{\widehat{g}} \ker \gamma \longrightarrow 0. \quad (3)$$

Next, β induces a map $\widehat{\beta} : \ker g' \circ \beta \longrightarrow \ker g'$. Using the fact that α is surjective we find $\beta(\ker g) = \beta(\operatorname{im} f) = \operatorname{im} \beta \circ f = \operatorname{im} f' \circ \alpha = \operatorname{im} f' = \ker g'$, hence $\operatorname{coker} \widehat{\beta} \circ \iota = 0$; applying the ring lemma to ι and $\widehat{\beta}$ and observing that $\ker \iota = 0$ we get the exact sequence

$$0 \longrightarrow \ker \widehat{\beta} \circ \iota \longrightarrow \ker \widehat{\beta} \longrightarrow \operatorname{coker} \iota \longrightarrow 0.$$

This gives (1) since $\ker \widehat{\beta} \circ \iota = \ker \beta \cap \ker g$, $\ker \widehat{\beta} = \ker \beta$, and finally $\operatorname{coker} \iota = \ker \gamma$ by (3).

The proof of the dual sequence (2) is similar. □